Exercise set 9 – Control – Solutions

Exercise 1

Consider a motor with a torque constant k_t, a reducer with a reduction ratio n = 32 and an incremental encoder with 500 quadrature increments (i.e. 500 x 4 pulses per revolution). The motor is tuned with a PD position controller at a sampling rate of 1kHz. J_m and J_l are the respective moments of inertia of the motor and of the driven load.

1. Give the resolution in position (in degrees) at the load level.
2. The speed is obtained by numerical differentiation. Give the resolution in speed if we calculate the derivative over:
   (a) one sampling period.
   (b) two sampling periods.
3. The backlash of the reducer is approximately 0.1°. Is this acceptable? If not, what can be done to solve the problem?
4. As a function of the controller gains K_p and T_d (according to the expressions in the course), determine the quantization noise on the control current.
5. Determine the smallest manageable displacement due to this calculated current resolution.
   Hint: Consider the relationship between input current and output position based on system dynamics.

Exercise 1 – Solution

1. The position resolution is $R(\theta) = \frac{360°}{500*4*32} = 0.005625°$
2. The derivation over x sampling periods (T_s) expresses the speed as:

$$\omega(k) = \frac{\theta(k) - \theta(k-x)}{xT_s}$$

   And the speed resolution when the derivation is done over x sampling periods is expressed as:

$$R(\omega) = \frac{R(\theta)}{xT_s}$$

   (a) Over one sampling period, we have $R(\omega)$ = 5.625°/s
   (b) Over two sampling periods, we have $R(\omega)$ = 2.812°/s

   The derivation over two sampling periods gives better speed resolution, the quantization noise due to the derivative is thus lower. This alternative is a good compromise when it is desired to reduce the quantization noise on the speed while maintaining a measurement of the position each sampling period.

3. The backlash of the reducer is very high compared to the position resolution. It is absolutely necessary to preconstrain the gear otherwise the speed quantization noise induced by this error linked to the backlash explodes.
4. The motor is controlled in current with a PD controller. The control current law is given by the following expression:

$$i = K_p e_\theta + K_d \frac{de_\theta(t)}{dt}$$

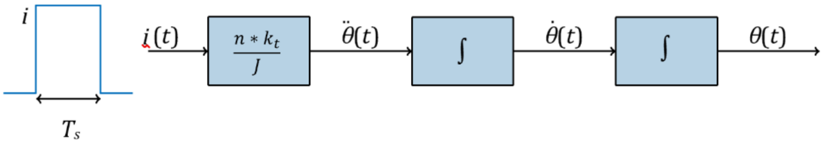
   K_p and K_d are the proportional and differential gains of the PD controller; e_θ is the position error.

   The smallest current induced by the quantization noise of the position and that of the speed is then given as follows:

$$i_q = K_p R(\theta) + K_d R(\omega) = K_p(R(\theta) + T_d R(\omega))$$

   <u>Remark</u>: The quantization noise is the digital resolution of the measured variable.

5. The smallest manageable displacement is that induced by a current input i_q, assuming this value is higher than the current resolution of the current amplifier. Knowing that the motor is in open loop over a sampling period T_s, the displacement induced by a current i_q corresponds to the value of the position for $t = T_s$:



$$\dot{\theta}(t) = i_q \frac{n\,k_t}{J} t$$

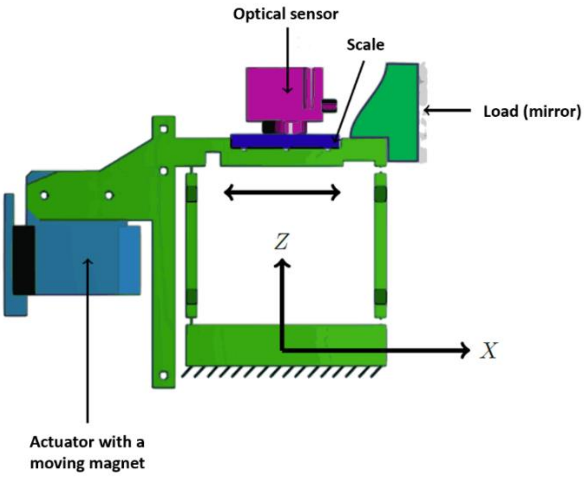
$$\Rightarrow \theta(t) = i_q \frac{n\,k_t}{J} \frac{t^2}{2}$$

J is the total moment of inertia at the load side $\Rightarrow J = J_l + n^2 J_m$

The smallest manageable displacement is $\theta(t = T_s)$, or $\theta_{min} = i_q \frac{n\,k_t}{2J} T_s^2$.

Exercise 2

Consider the following flexure hinge-based guide actuated by a current-controlled DC motor.

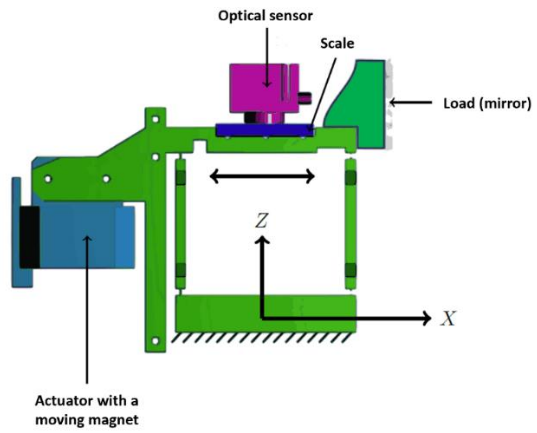


Let:

- k_f be the force constant between the active generated force and the motor current.
- m_g be the equivalent mass of the flexure hinge guide.
- m_l be the mass of the load (mirror and support).
- k_s be the equivalent stiffness of the leaf spring hinges.

1. (a) Give the transfer function current - displacement.
   (b) Determine the shape of the position response to a current step.
2. The actuator is looped with a PD controller. Make a block diagram of the closed loop with this controller.
3. (a) Give the expression of the new closed loop transfer function.
   (b) Identify the meaning of each coefficient of this transfer function. (We'll consider a step input – we want to reach a constant desired position x_d, i.e. $\frac{dx_d}{dt} = 0$)

Exercise 2 – Solution



The flexure hinge guide functions as a spring free in the direction X and rigid in the other directions. By fixing the origin of the axis X at the "zero" point of the spring, the Newton law applied to the movement is written as follows:

$$\sum F = F_m - k_s x = (m_g + m_l)\ddot{x}$$
$$\Rightarrow k_f i - k_s x = (m_g + m_l)\ddot{x}$$

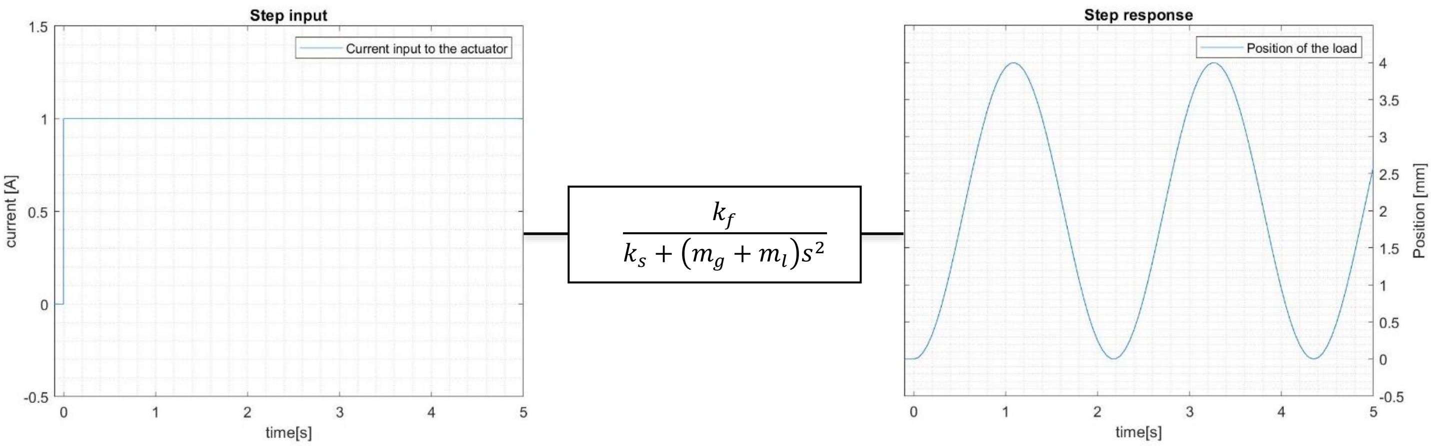
1. (a) The dynamic model is given as:

$$k_f i = (m_g + m_l)\ddot{x} + k_s x$$

The transfer function is therefore:

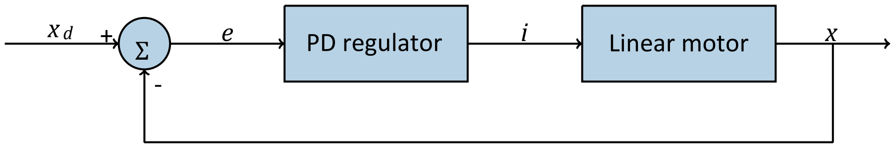
$$\frac{X}{I} = \frac{k_f}{k_s + (m_g + m_l)s^2}$$

(b) This transfer function is of type $\frac{G}{s^2 + b^2}$. It describes a purely oscillatory system. The answer is therefore a sine, which is explained by the absence of damping. Below is a representation with: k_f= 1 [N/A], k_s = 0.5[N/mm], m_g= 50[g], m_l= 10g.



For this simulation please refer to attached MATLAB script.

2. PD controller:



3. (a) The dynamic model is:

$$k_f i = (m_g + m_l)\ddot{x} + k_s x$$

Therefore, in closed loop:

$$k_f\left(K_p(x_d - x) + K_p T_d \frac{dx_d}{dt} - K_p T_d \frac{dx}{dt}\right) = (m_g + m_l)\ddot{x} + k_s x$$

To identify the transfer function, we take the Laplace transform:

$$k_f\left(K_p(X_d - X) + K_p T_d X_d s - K_p T_d X s\right) = (m_g + m_l) X s^2 + k_s X$$

$$\Rightarrow G(s) = \frac{X}{X_d} = \frac{k_f K_p (1 + T_d s)}{(m_g + m_l)s^2 + (k_f K_p T_d)s + (k_f K_p + k_s)}$$

(b) First, let's compute the static gain (step input).

$$G(0) = \frac{k_f K_p}{k_f K_p + k_s}$$

It is not unitary, which means that there is a steady state error. It can be reduced by increasing the stiffness of the controller and canceled when it is infinitely high:

$$\frac{k_f K_p}{k_f K_p + k_s} = \frac{k_f K_p}{k_f K_p \left\{1 + \frac{k_s}{k_f K_p}\right\}} = \frac{1}{1 + \frac{k_s}{k_f K_p}}$$

$$\Rightarrow \lim_{K_p \to \infty} \frac{k_f K_p}{k_f K_p + k_s} = \frac{1}{1 + \lim_{K_p \to \infty} \frac{k_s}{k_f K_p}} = \frac{1}{1 + 0} = 1$$

The numerator is typical of a PD controller. The presence of a s term means that the system will have a derivative action. It will improve the system responsiveness and potentially the stability, but may also increase the sensitivity to high frequency noise and, thus, also potentially make the system unstable. Looking at the denominator, the term in s is the damping term which stabilizes the positioning system. Finally, the term in s^2 is the oscillatory term, it is linked to the inertia of the system.